Mimetic Discrete Models with Weak Material Laws,
or
Least Squares Principles Revisited.

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The Plan

Part I
Mimetic Methods

1. What is a mimetic discretization
2. An algebraic topology framework
3. Direct and conforming discretizations

Mixed, Galerkin and Least-Squares methods for 2nd order problems share a common ancestor: the 4-field principle

A new interpretation of Least Squares: Realizations of a weak discrete Hodge * operator

A prelude:
Least-Squares Principles

What are least-squares and the reasons to use them

LS acquire surprising new properties when elements from mixed methods are used

For diffusion problems they give
- the same **scalar** as **Galerkin** method
- the same **flux** as in the **mixed** method

Part II
Compatibility matters!

Mimetic LSP for eddy currents and diffusion/heat equations and their advantages over nodal LS.

Mac Hyman, Misha Shashkov
T-7
Los Alamos National Laboratory

Max Gunzburger
CSIT
Florida State University

A Prelude

Least-squares 101

$\mathcal{L}u = f$ in Ω

$\mathcal{R}u = h$ on Γ

$$\min_{u\in X} J(u;f,h) \equiv \frac{1}{2}\left(\|\mathcal{L}u - f\|^2_{X,\Omega} + \|\mathcal{R}u - h\|^2_{Y,\Gamma}\right)$$

$$(\mathcal{L}u,\mathcal{L}v)_\Omega + (\mathcal{R}u,\mathcal{R}v)_\Gamma = (f,\mathcal{L}u)_\Omega + (h,\mathcal{R}v)_\Gamma$$

$$\mathbf{Au} = \mathbf{b}$$

Top 3 reasons people

want to do least squares:

☺ **Using C^0 nodal elements**

☺ **Avoiding inf-sup conditions**

☺ **Solving SPD systems**

don't want to do least squares:

☹ **Conservation**

☹ **Conservation**

☹ **Conservation**

We will show that:

- Using **nodal elements** is not necessarily the best choice in LSFEM, and so it is arguably the **least-important advantage** attributed to least-squares methods
- By using **other** elements least-squares acquire **additional conservation** properties
- Surprisingly, this kind of least-squares turns out to be **related** to **mixed methods**

Introduced by Jespersen (1977), Fix, Gunzburger and Nicolaides (1977-85). See also Cai, Carey, Chang, Jiang, Lazarov, Manteuffel et al (1994-2000) and the survey B. & Gunzburger in SIAM Review, 1998

Least-squares for diffusion

$$\left.\begin{aligned}\nabla\cdot\mathbf{u}+\gamma\phi&=f\\ \mathbf{u}+\nabla\phi&=0\end{aligned}\right\} \Leftrightarrow J(\mathbf{u},\phi;f)=\frac{1}{2}\left(\|\nabla\cdot\mathbf{u}+\gamma\phi-f\|_0^2+\|\mathbf{u}+\nabla\phi\|_0^2\right)=0$$

$$-\nabla\cdot\nabla\phi+\gamma\phi=f \qquad \Longleftrightarrow \qquad \min_{\mathbf{v}\in H_N(\Omega,div);\psi\in H_D^1(\Omega)} J(\mathbf{v},\psi;f)$$

$$(\nabla\cdot\mathbf{u}+\gamma\phi,\nabla\cdot\mathbf{v})+(\mathbf{u}+\nabla\phi,\mathbf{v})=(f,\nabla\cdot\mathbf{v}) \quad \forall\mathbf{v}\in H_N(\Omega,\text{div})$$

$$(\nabla\cdot\mathbf{u}+\gamma\phi,\gamma\psi)+(\mathbf{u}+\nabla\phi,\nabla\psi)=(f,\gamma\psi) \quad \forall\psi\in H_D(\Omega,\text{grad})$$

"Artificial" energy norm $J(\mathbf{u},\phi;0)=\frac{1}{2}\left(\|\nabla\cdot\mathbf{u}+\gamma\phi\|_0^2+\|\mathbf{u}+\nabla\phi\|_0^2\right)=|||(\mathbf{u},\phi)|||^2$

Norm equivalence $C_1\left(\|\mathbf{u}\|_{div}^2+\|\phi\|_1^2\right)\le|||(\mathbf{u},\phi)|||^2\le C_2\left(\|\mathbf{u}\|_{div}^2+\|\phi\|_1^2\right)$

Inner-product equivalence $Q_{LS}(\mathbf{u},\phi;\mathbf{v},\psi)=(\nabla\cdot\mathbf{u}+\gamma\phi,\nabla\cdot\mathbf{v}+\gamma\psi)+(\mathbf{u}+\nabla\phi,\mathbf{v}+\nabla\psi)$

Stability

$C_1\left(\|\mathbf{u}\|_{div}^2+\|\phi\|_1^2\right)\le Q_{LS}(\mathbf{u},\phi;\mathbf{u},\phi)$ ← **coercivity**

continuity → $Q_{LS}(\mathbf{u},\phi;\mathbf{v},\psi)\le C_2\left(\|\mathbf{u}\|_{div}^2+\|\phi\|_1^2\right)^{1/2}\left(\|\mathbf{v}\|_{div}^2+\|\psi\|_1^2\right)^{1/2}$

In the dark ages least-squares were deemed immune to compatibility

Discrete equations $$Q_{LS}(\mathbf{u}_h,\phi_h;\mathbf{v}_h,\psi_h)=(f,\nabla\cdot\mathbf{v}_h+\gamma\psi_h)\quad\forall(\mathbf{v}_h,\psi_h)\in\mathbf{V}_h\times S_h$$

Coercivity is inherited on all closed subspaces, and so any

$$\mathbf{V}_h\subset H(\Omega,div)\quad\&\quad S_h\subset H^1(\Omega)$$ (including C^0)

are **sufficient** for **stability** of LSFEM and **quasi-optimal** energy norm error estimates

This was deemed to be a "**get out of jail**" card needed to throw away compatibility

$\Rightarrow$ all variables "**can**" be approximated by **the same, equal order** C^0 spaces

But:

$$\underbrace{\|\mathbf{u}-\mathbf{u}_h\|_{div}+\|\phi-\phi_h\|_1}_{\text{energy norm}}\le C\inf_{(\mathbf{v}_h,\psi_h)\in\mathbf{V}_h\times S_h}\|\mathbf{u}-\mathbf{v}_h\|_{div}+\|\phi-\phi_h\|_1$$

$$\|\phi-\phi_h\|_0\le Ch\|\phi-\phi_h\|_1$$ $\leftarrow$ Using duality

There was a little problem…

For LSP: conformity $\Rightarrow$ stability but conformity $\not\Rightarrow$ optimal L^2 accuracy!

$\mathbf{V}_h \subset H(\Omega, div)$ & $S_h \subset H^1(\Omega)$ **is insufficient** for optimal L^2 convergence of $\boldsymbol{v}_h$!

LS vs BA	scalar		vector	
	L^2	H^1	L^2	$H(div)$
P1	2.00	1.00	1.38	0.99
BA	2.00	1.00	2.00	1.00
P2	3.00	2.00	2.02	2.00
BA	3.00	2.00	3.00	2.00

Optimal convergence of v_h in L^2 has been achieved in 2 ways

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(Carey et al, Jiang, Manteuffel et al. 1994-1997)

By using an augmented LS principle

Idea $\quad \mathbf{u}+\nabla\phi=0 \quad \Rightarrow \quad \nabla\times\mathbf{u}=0$

Augmented PDE $\quad \begin{Bmatrix} \nabla\cdot\mathbf{u}+\gamma\phi=f \\ \mathbf{u}+\nabla\phi=0 \end{Bmatrix} \,\&\, \nabla\times\mathbf{u}=0$ in Ω; $\quad \phi=0$ on Γ_D; $\mathbf{u}\cdot\mathbf{n}=0$ on Γ_N

Functional $\quad J(\mathbf{u},\phi;f)=\frac{1}{2}\left(\|\nabla\cdot\mathbf{u}+\gamma\phi-f\|_0^2+\|\mathbf{u}+\nabla\phi\|_0^2+\|\nabla\times\mathbf{u}\|_0^2\right)$

Norm equivalence $\quad \boxed{C_1\left(\|\mathbf{u}\|_1^2+\|\phi\|_1^2\right)\le |||\,\mathbf{u},\phi\,|||\le C_2\left(\|\mathbf{u}\|_1^2+\|\phi\|_1^2\right)}$

Error estimate (P2) $\quad \rightarrow$

$$\|\mathbf{u}-\mathbf{u}_h\|_1+\|\phi-\phi_h\|_1\le Ch^2\left(\|\mathbf{u}\|_3+\|\phi\|_3\right)$$

$$\|\mathbf{u}-\mathbf{u}_h\|_0+\|\phi-\phi_h\|_0\le Ch^3\left(\|\mathbf{u}\|_3+\|\phi\|_3\right)$$

The trouble with this approach

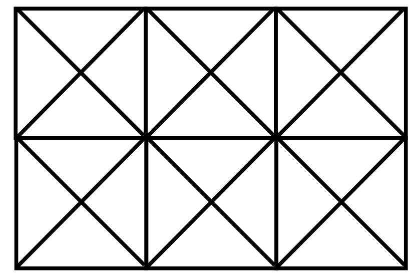
The range of the solution operator is restricted to a "**smoother**" space, causing the least-squares principle to **miss** less regular solutions that are **admissible** for the original PDE! **We will see an example of this problem.**

Or, by using a special grid

The Grid Decomposition Property (GDP)

Fix, Gunzburger, Nicolaides, 1976

$$\forall \mathbf{v}_h \in V_h \quad \begin{cases} \mathbf{v}_h = \mathbf{w}_h + \mathbf{z}_h \\ \nabla \cdot \mathbf{z}_h = 0 \end{cases} \qquad \begin{cases} (\mathbf{w}_h, \mathbf{z}_h) = 0 \\ \|\mathbf{w}_h\|_0 \le C\left(\|\nabla \cdot \mathbf{v}_h\|_{-1} + h\|\nabla \cdot \mathbf{v}_h\|_0\right) \end{cases}$$



The (only known) C^0 example

Theorem

GDP is *necessary* and *sufficient* for *stable* and *optimally accurate* mixed discretization of the **Least-Squares Principle (and the Mixed Method)**

Fix, Gunzburger, Nicolaides, Comp. Math with Appl. 5, pp.87-98, 1979

Using the criss-cross grid and $S_h = \nabla \bullet \mathbf{V}_h$:

$$\|\mathbf{u} - \mathbf{u}_h\|_{div} + \|\phi - \phi_h\|_1 \le Ch^1\left(\|\mathbf{u}\|_2 + \|\phi\|_2\right)$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + \|\phi - \phi_h\|_0 \le Ch^2\left(\|\mathbf{u}\|_2 + \|\phi\|_2\right)$$

The mixed Galerkin connection

(Bochev, Gunzburger, SINUM 2005)

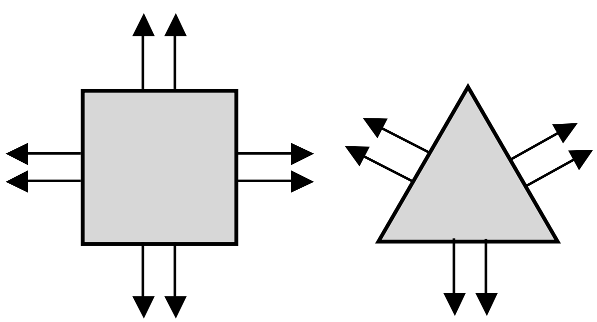
Lemma

$(\mathbf{V}_h, S_h)$ satisfies the inf-sup condition $\Rightarrow$ $\mathbf{V}_h$ verifies GDP

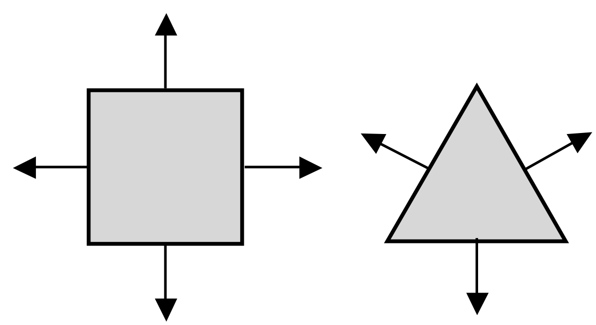
There are plenty of spaces that verify GDP

Except that they are **not C^0** (nodal)!

BDM(k) spaces k≥1



RT(k) spaces k≥0



"Well-done" (mimetic) least-squares

Using nodal C^0 elements for all variables is not the best choice!

(despite of what some people tell you!)

Instead, pose the discrete LSP $\min_{\mathbf{v}_h \in D^h; \psi_h \in G^h} J(\mathbf{v}_h, \psi_h; f)$ **on this pair of spaces:**

$$D^h \subset H_N(\Omega, \text{div}) \quad \rightarrow \quad \text{any with GDP}$$

$$G^h \subset H^1_D(\Omega, \text{grad}) \quad \rightarrow \quad \text{any that is } C^0$$

Theorem. *For proof see Bochev, Gunzburger,* ***SIAM J. NUM. ANAL. 2005***

For $\phi_h \in P_k$ and $\mathbf{u}_h \in BDM_k$:	For $\phi_h \in P_k$ and $\mathbf{u}_h \in RT_k$:
$\lVert\phi - \phi_h\rVert_0 + \lVert\mathbf{u} - \mathbf{u}_h\rVert_0 = O(h^{k+1})$	$\lVert\phi - \phi_h\rVert_0 + \lVert\mathbf{u} - \mathbf{u}_h\rVert_0 = O(h^{k})$
$\lVert\phi - \phi_h\rVert_1 + \lVert\mathbf{u} - \mathbf{u}_h\rVert_{div} = O(h^{k})$	$\lVert\phi - \phi_h\rVert_1 + \lVert\mathbf{u} - \mathbf{u}_h\rVert_{div} = O(h^{k})$

Velocity and pressure spaces need not form a <u>stable mixed</u> pair!

A startling property of mimetic LS

Theorem

Assume that $(\phi^h, \mathbf{u}^h)$ solves the minimization problem

$$\min_{\phi^h \in G^h;\, \mathbf{u}^h \in D^h} \tilde{K}(\phi^h, \mathbf{u}^h) \equiv \frac{1}{2}\left(\left\| \mathbf{A}^{-1/2}\left(\mathbf{u}^h + \mathbf{A}\nabla\phi^h\right)\right\|_0^2 + \left\|\gamma^{-1/2}\left(\nabla\cdot\mathbf{u}^h + \gamma\phi^h - f\right)\right\|_0^2 \right)$$

if $\gamma>0$, $(\phi^h, \mathbf{u}^h)$ is **conservative** in the sense that there exists $\mathbf{w}^h \in C^h;\ \psi^h \in Q^h$ such that

- $(\phi^h, \mathbf{w}^h) \in G^h \times C^h$ solves the Ritz-Galerkin method and $\nabla\phi^h + \mathbf{w}^h = 0$
- $(\psi^h, \mathbf{u}^h) \in Q^h \times D^h$ solves the Mixed Galerkin method and $\nabla\cdot\mathbf{u}^h + \gamma\psi^h = \Pi^h f$

In other words, the **mimetic least-squares** method computes

The same scalar approximation as in the Ritz-Galerkin method
The same vector approximation as in the Mixed Galerkin method

Mimetic LS = Galerkin + Mixed Galerkin

error \ grid		16	32	64	128
L2 $\mathbf{u}$	Mimetic LS	0.1514803E+00	0.7192623E-01	0.3523105E-01	0.1745720E-01
	Mixed	0.1514803E+00	0.7192623E-01	0.3523105E-01	0.1745720E-01
H(div)	Mimetic LS	0.2869324E+01	0.1397179E+01	0.6894290E+00	0.3426716E+00
	Mixed	0.2869324E+01	0.1397179E+01	0.6894290E+00	0.3426716E+00
L2 ϕ	Galerkin	0.3997943E-02	0.9378368E-03	0.2274961E-03	0.5621838E-04
	Mimetic LS	0.3997943E-02	0.9378368E-03	0.2274961E-03	0.5621838E-04
	Mixed	0.3679584E-01	0.1778803E-01	0.8750616E-02	0.4340574E-02
H1 ϕ	Mimetic LS	0.2671283E+00	0.1296329E+00	0.6383042E-01	0.3166902E-01
	Galerkin	0.2671283E+00	0.1296329E+00	0.6383042E-01	0.3166902E-01

Scalar: L2 and H1 errors of Mimetic LS and Galerkin identical

Vector: L2 and H(div) errors of Mimetic LS and Mixed Galerkin identical

A $64K Question

We see that a Least Squares perform better when using

– nodal C^0 space for the **scalar** (same as in the **Galerkin** FEM)

– H(div) conforming space for the **vector** (same as in the **Mixed Galerkin** FEM)

Q: what are the fundamental reasons for the method to acquire these new and attractive properties?

To answer this question we will use algebraic topology to develop a framework for compatible PDE discretizations. Then, we will examine different discrete models arising from this framework.

Part I

Algebraic topology approach

Algebraic topology provides the tools to mimic the PDE structure

- **Computational grid** is algebraic topological complex
- ***k*-forms** are encoded as *k*-cell quantities (*k*-cochains)
- **Derivative** is provided by the coboundary
- **Inner product** induces combinatorial Hodge theory
- **Singular cohomology** preserved by the complex

Framework for mimetic discretizations (*Bochev, Hyman, IMA Proceedings)*

- **Translation:** Fields → forms → cochains
- **Basic mappings:** **reduction** and **reconstruction**
  - **Combinatorial operations:** induced by **reduction** map
  - **Natural operations:** induced by **reconstruction** map
  - **Derived operations:** induced by **natural** operations

Branin (1966), Dodzuik (1976), Hyman & Scovel (1988-92), Nicolaides (1993), Dezin (1995), Shashkov (1990-), Mattiussi (1997), Schwalm (1999), Teixeira (2001), Marsden et al (DEC) and many others…

Differential Forms

Smooth differential forms	$\Lambda^k(\Omega): \quad x \to \omega(x) \in \Lambda^k(T_x\Omega)$
DeRham complex	$\mathbf{R} \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \to 0$
Metric conjugation	$*: \Lambda^k(T_x\Omega) \to \Lambda^{n-k}(T_x\Omega) \Leftrightarrow \omega \wedge *\xi = (\omega,\xi)_x \omega_n$
L² inner product on Λ*k*(Ω)	$(\omega,\xi)_\Omega = \int_\Omega (\omega,\xi)_x \omega_n \quad \Rightarrow \quad (\omega,\xi)_\Omega = \int_\Omega \omega \wedge *\xi$
Codifferential	$d^*: \Lambda^{k+1}(\Omega) \to \Lambda^{k+1}(\Omega) \quad \Leftrightarrow \quad (d\omega,\xi)_\Omega = (\omega, d^*\xi)_\Omega$
Hodge Laplacian	$\Delta: \Lambda^k(\Omega) \to \Lambda^k(\Omega) \quad \to \quad \Delta = dd^* + d^*d$
Completion of Λ*k*(Ω)	$\Lambda^k(L^2,\Omega)$
Sobolev spaces	$\Lambda^k(d,\Omega) = \{\omega \in \Lambda^k(L^2,\Omega) \mid d\omega \in \Lambda^{k+1}(L^2,\Omega)\}$

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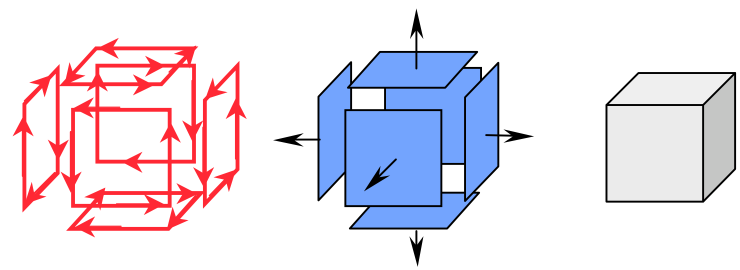
Chains and cochains

Computational grid = Chain complex

$$\partial : C_k \to C_{k-1}$$

$$\partial\partial = 0 \qquad C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} C_3$$



$$0 = \partial\partial K^3 \xleftarrow{\partial} \partial K^3 \xleftarrow{\partial} K^3$$

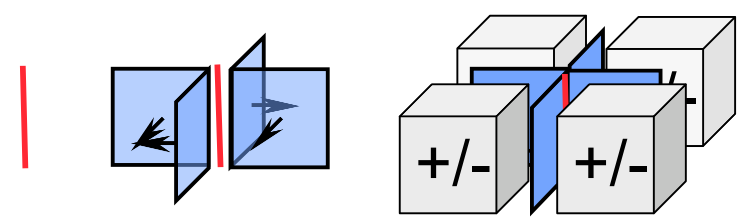
Field representation = Cochain complex

$$C^k = L(C_k, \mathbf{R}) = C_k^{\ *} \qquad \langle \sigma^i, \sigma_j \rangle = \delta_{ij}$$

$$\delta : C^k \to C^{k+1} \qquad \langle \omega, \partial\eta \rangle = \langle \delta\omega, \eta \rangle$$

$$\delta\delta = 0 \qquad C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} C^2 \xrightarrow{\delta} C^3$$

$$K^1 \xrightarrow{\delta} \delta K^1 \xrightarrow{\delta} \delta\delta K^1 = 0$$



Basic mappings

Reduction

$$\mathcal{R} : \Lambda^k\left(L^2,\Omega\right) \to C^k$$

Natural choice

$$\langle \mathcal{R}\omega,\sigma\rangle = \int_\sigma \omega$$

DeRham map

$$\mathcal{R}d = \delta\mathcal{R}$$

Proof

$$\langle \delta\mathcal{R}\omega,c\rangle = \langle \mathcal{R}\omega,\partial c\rangle = \int_{\partial c}\omega = \int_c d\omega = \langle \mathcal{R}d\omega,c\rangle$$

$$\begin{tikzcd}
\cdots \Lambda^k \arrow[r, "d"] \arrow[d] & \Lambda^{k+1}\cdots \arrow[d] \\
\cdots C_k & C_{k+1}\cdots \arrow[l, "\partial"'] \\
\cdots C^k \arrow[r, "\delta"] \arrow[u, leftrightarrow] & C^{k+1}\cdots \arrow[u, leftrightarrow]
\end{tikzcd}$$

$$\begin{tikzcd}
\Lambda^k \arrow[r, "d"] \arrow[d, "\mathcal{R}"'] \arrow[dr, phantom, "\textbf{CDP I}"] & \Lambda^{k+1} \arrow[d, "\mathcal{R}"] \\
C^k \arrow[r, "\delta"] & C^{k+1}
\end{tikzcd}$$

natural

$$\begin{tikzcd}
\Lambda^k \arrow[r, "d"] & \Lambda^{k+1} \\
C^k \arrow[r, "\delta"] \arrow[u, "\mathcal{I}"] \arrow[ur, phantom, "\textbf{CDP2}"] & C^{k+1} \arrow[u, "\mathcal{I}"']
\end{tikzcd}$$

required

$$\text{Range } \mathcal{IR} = \Lambda^k\left(L^2,K\right) \subset \Lambda^k\left(L^2,\Omega\right)$$

$$\text{Range } \mathcal{IR} = \Lambda^k\left(d,K\right) \subset \Lambda^k\left(d,\Omega\right)$$

Reconstruction

$$\mathcal{I} : C^k \to \Lambda^k\left(L^2,\Omega\right)$$

No natural choice

$$\mathcal{RI} = id$$

$$\mathcal{IR} = id + O(h^s)$$

$$\ker \mathcal{I} = 0$$

Conforming

$$\mathcal{I} : C^k \to \Lambda^k\left(d,\Omega\right)$$

$$\mathcal{I}d = \delta\mathcal{I}$$

Combinatorial operations

Discrete derivative

Forms are dual to **manifolds** $\langle d\omega,\Omega\rangle = \langle \omega,\partial\Omega\rangle$

Cochains are dual to **chains** $\langle \delta a,\sigma\rangle = \langle a,\partial\sigma\rangle$

δ approximates *d* on cochains

Discrete integral

$$\int_\sigma a = \langle a,\sigma\rangle$$

Stokes theorem

$$\langle \delta a,\sigma\rangle = \langle a,\partial\sigma\rangle$$

grad

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

curl

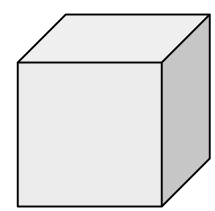
$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

div

$$\begin{pmatrix} -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\delta\delta = 0$$

$$\mathcal{R}d = \delta\mathcal{R}$$



Natural and derived operations

Natural Inner product $(a,b)_x = (\mathcal{I}a, \mathcal{I}b)_x$ $\quad (a,b)_\Omega = \int_\Omega (a,b)_x \omega_n = (\mathcal{I}a, \mathcal{I}b)_\Omega$

Wedge product $\wedge : C^k \times C^l \mapsto C^{k+l}$ $\quad a \wedge b = \mathcal{R}(\mathcal{I}a \wedge \mathcal{I}b)$

Derived Adjoint derivative $\delta^* : C^{k+1} \mapsto C^k$ $\quad (\delta^* a, b)_\Omega = (a, \delta b)_\Omega$

Provides a second set of **grad**, **div** and **curl** operators. Scalars encoded as 0 or 3-forms, vectors as 1 or 2-forms, derivative choice depends on encoding.

Discrete Laplacian $D : C^k \mapsto C^k$ $\quad D = \delta^* \delta + \delta \delta^*$

Derived operations are necessary to avoid **internal inconsistencies** between the discrete operations: $\mathcal{I}$ is only **approximate inverse** of $\mathcal{R}$ and natural operations will clash

Example Natural adjoint $d^* = (-1)^k * d *$ $\longrightarrow$ $\delta^* = (-1)^k \mathcal{R} * d * \mathcal{I}$

$\mathcal{I}$ must be regular and $(\delta^* a, b)_\Omega = (a, \delta b)_\Omega + O(h^s)$ $\Rightarrow$ δ^* not true adjoint

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Mimetic properties (I)

Discrete Poincare lemma (existence of potentials in contractible domains)

$$d\omega_k = 0 \quad \Rightarrow \quad \omega_k = d\omega_{k+1} \qquad\qquad \delta c^k = 0 \quad \Rightarrow \quad c^k = \delta c^{k+1}$$

Discrete Stokes Theorem

$$\left\langle d\omega_{k-1}, c_k \right\rangle = \left\langle \omega_{k-1}, \partial c_k \right\rangle \qquad\qquad \left\langle \delta c^{k-1}, c_k \right\rangle = \left\langle c^{k-1}, \partial c_k \right\rangle$$

Discrete "Vector Calculus"

$$dd = 0 \qquad\qquad \delta\delta = \delta * \delta * = 0$$

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega \qquad\qquad a \wedge b = (-1)^{kl} b \wedge a$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \qquad\qquad \delta(a \wedge b) = \delta a \wedge b + (-1)^k a \wedge \delta b$$

(Regular $\mathcal{I}$)

Features of the continuum system that is implied by differential forms calculus is inherited by the discrete model

Called ***mimetic*** property by Hyman and Scovel (1988)

Mimetic properties (II)

Inner product induces combinatorial Hodge theory on cochains

$$\text{Co-cycles of } (\Lambda^0, \Lambda^1, \Lambda^2, \Lambda^3) \xrightarrow{\mathcal{R}} \text{co-cycles of } (C^0, C^1, C^2, C^3)$$

$$d\omega = 0 \quad \Rightarrow \quad \delta \mathcal{R}\omega = 0$$

Discrete Harmonic forms

$$H^k(\Omega) = \left\{ \eta \in \Lambda^k(\Omega) \mid d\eta = d^*\eta = 0 \right\} \qquad H^k(K) = \left\{ c^k \in C^k \mid \delta c^k = \delta^* c^k = 0 \right\}$$

Discrete Hodge decomposition

$$\omega = d\rho + \eta + d^*\sigma$$

GDP

Theorem $\quad \dim\ker(\Delta) = \dim\ker(D)$

Remarkable property of the mimetic *D* - kernel size is a **topological invariant!**

Discrete $*$ operation

Natural definition (Bossavit)

$$*_N : C^k \mapsto C^{n-k} \qquad *_N = \mathcal{R} * \mathcal{I}$$

Derived definition (Hiptmair)

$$*_D : C^k \mapsto C^{n-k} \qquad \int_\Omega a \wedge *_D b = (a,b)_\Omega$$ mimics $$(\omega,\xi)_\Omega = \int_\Omega \omega \wedge *\xi$$

Theorem

$$*_N \mathcal{R}\omega^h = \mathcal{R} * \omega^h \quad \forall \omega^h \in \mathrm{Range}(\mathcal{I}\mathcal{R})$$ CDP on the range

$$\int_\Omega \mathcal{I}\mathcal{R}(\mathcal{I}a \wedge \mathcal{I} *_D b) = \int_\Omega (\mathcal{I}a \wedge *\mathcal{I}b)$$ Weak CDP

$$\int_\Omega b \wedge *_N b = (a,b)_\Omega + O(h^s) \qquad *_N = *_D + O(h^s)$$

The trouble with the discrete $*$

Action of $*$ must be coordinated with the other discrete operations

	$(\bullet,\bullet)$	$\wedge$	δ^*	$\mathcal{R}$	$\mathcal{I}$
$*_N$	—	—	—	✓	—
$*_D$	✓	✓	—	—	—

Analytic $*$ is a **local, invertible** operation $\Rightarrow$ **positive diagonal** matrix

$\dim C^k \neq \dim C^{n-k}$ $\Rightarrow$ $*_N : C^k \mapsto C^{n-k}$ **cannot be a square matrix!**

Construction of $*$ is nontrivial task unless a primal-dual grid is used!

Implications

A consistent discrete framework requires a choice of a primary operation
either $*$ or $(\cdot,\cdot)$ but not both

A **discrete $*$** is the primary concept in Hiptmair (2000), Bossavit (1999)

– Inner product derived from discrete $*$
– discrete $*$ used in **explicit discretization of material laws**

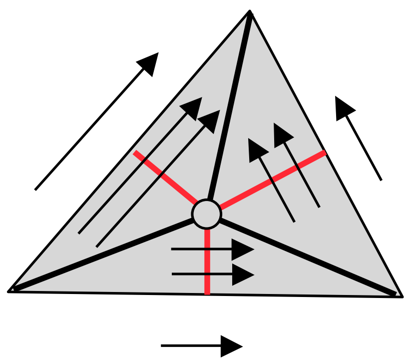
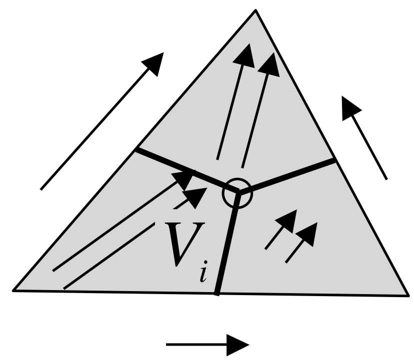
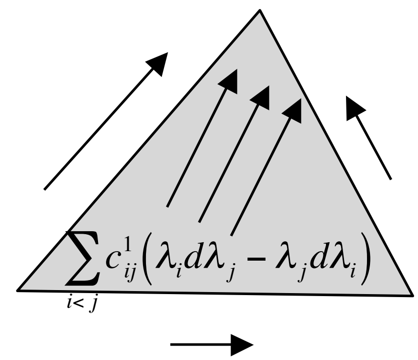
The **natural inner product** is the primary operation in our approach

– **Sufficient** to give rise to combinatorial Hodge theory on cochains
– **Easier** to define than a discrete $*$ operation
– **Incorporate** material laws in the natural inner product, or
– **Enforce** material laws **weakly** (justified by their approximate nature)

Algebraic equivalents

Operation	Matrix form	type
δ	$\mathbf{D}_k$	{-1,0,1}
$(\cdot,\cdot)$	$\mathbf{M}_k$	SPD
$a_1 \wedge b_1$	$\sum \mathbf{W}_{11}$	Skew symm.
$a_1 \wedge b_2$	$\sum \mathbf{W}_{12}$	$\mathbf{W}_{12}^{T}=\mathbf{W}_{21}$
$b_2 \wedge a_1$	$\sum \mathbf{W}_{21}$	
δ^*	$\mathbf{M}_k^{-1}\, \mathbf{D}_k^{T}\mathbf{M}_{k+1}$	rectangular
$\mathcal{D}$	$\mathbf{M}_k^{-1}\, \mathbf{D}_k^{T}\mathbf{M}_{k+1}\mathbf{D}_k + \mathbf{D}_{k-1}\mathbf{M}_{k-1}^{-1}\, \mathbf{D}_{k-1}^{T}\mathbf{M}_k$	square
$*_D$	$\mathbf{W}_{12}(*_D \mathbf{a})=\mathbf{M}_3\mathbf{a}$	pair

Reconstruction and natural inner products

	Co-volume	Mimetic	Whitney
$\mathcal{I}$			
	Nicolaides, Trapp (1992-04)	*Hyman, Shashkov, Steinberg (1985-04)*	*Dodzuik (1976) Hyman, Scovel (1988)*

$$\omega_{ij}^1 = \lambda_i d\lambda_j - \lambda_j d\lambda_i$$

M

$$\begin{pmatrix} h_1 h_1^\perp & & \\ & h_2 h_2^\perp & \\ & & h_3 h_3^\perp \end{pmatrix}$$

$$\begin{pmatrix} \frac{V_2}{\sin^2\phi_2}+\frac{V_3}{\sin^2\phi_3} & \frac{V_3\cos\phi_3}{\sin^2\phi_3} & \frac{V_2\cos\phi_2}{\sin^2\phi_2} \\ \frac{V_3\cos\phi_3}{\sin^2\phi_3} & \frac{V_1}{\sin^2\phi_1}+\frac{V_3}{\sin^2\phi_3} & \frac{V_1\cos\phi_1}{\sin^2\phi_1} \\ \frac{V_2\cos\phi_2}{\sin^2\phi_2} & \frac{V_1\cos\phi_1}{\sin^2\phi_1} & \frac{V_1}{\sin^2\phi_1}+\frac{V_2}{\sin^2\phi_2} \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & (w_{ij}, w_{kl}) & \\ \cdots & \cdots & \cdots \end{pmatrix}$$

δ^* — **local** | **non-local** | **non-local conforming**

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Mimetic discretization of magnetic diffusion: translation to forms

1st order PDE with material laws

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \mathbf{J} = \sigma \mathbf{E}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad \mathbf{B} = \mu \mathbf{H}$$

$$de = -d_t b \qquad *_{\sigma^{-1}} j = e$$

$$dh = j \qquad *_{\mu^{-1}} b = h$$

primal → (e,b)

dual → (h,j)

domain curl range

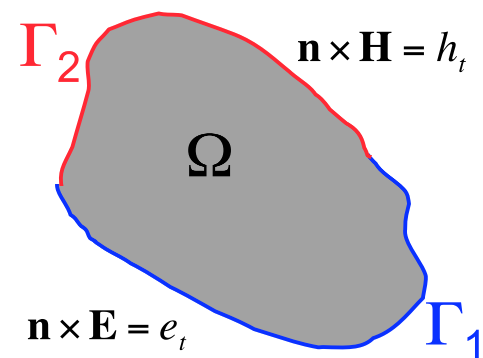
1st order PDE with codifferentials

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\sigma} \nabla \times \frac{1}{\mu} \mathbf{B} = \mathbf{E}$$

$$de = -d_t b$$

$$e = *_{\sigma^{-1}} d *_{\mu^{-1}} b$$



2nd order PDE

$$\nabla \times \frac{1}{\sigma} \nabla \times \frac{1}{\mu} \mathbf{B} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$d *_{\sigma^{-1}} d *_{\mu^{-1}} b = -d_t b$$

NOTE: we could have eliminated the primal pair (E,B) and obtain the last two equations in terms of the dual pair (H,J).

Option (I): Material properties via codifferentials

$$\delta\delta^* b^2 = -\delta_t b^2$$

$$e^1 \in C^1; \quad b^2 \in C^2$$

$$\delta e^1 = -\delta_t b^2$$
$$e^1 = \delta^* b^2$$

$$d *_{\sigma^{-1}} d *_{\mu^{-1}} b = -d_t b$$

Direct

Conforming

$$de = -d_t b$$
$$e = *_{\sigma^{-1}} d *_{\mu^{-1}} b$$

$$\left(db_h^2, d\hat{b}_h^2\right)_\Omega = \left(-d_t b_h^2, \hat{b}_h^2\right)_\Omega$$

$$e_h^1 \in \Lambda^1(d,K); \quad b_h^2 \in \Lambda^2(d,K)$$

$$de_h^1 = -d_t b_h^2$$
$$\left(e_h^1, \hat{e}_h^1\right)_\Omega = \left(b_h^2, d\hat{e}_h^1\right)_\Omega$$

Theorem (Bochev & Hyman)

Assume that $\mathcal{I}$ is **conforming** reconstruction operator. Then, the **direct** and the **conforming** mimetic methods are completely equivalent.

Option (II)
Mimetic models with weak material laws

Translate 1st order system to an equivalent 4-field constrained optimization problem

$$\begin{aligned} de &= -d_t b \qquad *_{\sigma^{-1}} j = e \\ dh &= j \qquad\quad *_{\mu^{-1}} b = h \end{aligned} \quad \Longrightarrow \quad \min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(*_{\sigma^{-1}} j - e\right)\right\|^2 + \left\| \sqrt{\mu}\left(*_{\mu^{-1}} b - h\right)\right\|^2 \right)$$

subject to $de = -d_t b$ and $dh = j$

Discretize in time

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(*_{\sigma^{-1}} j - e\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left(*_{\mu^{-1}} b - h\right)\right\|^2 \right)$$ subject to $de = -\gamma\left(b - \bar{b}\right)$ and $dh = j$

Discretize in space (fully mimetic)

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} j_h^2 - e_h^1\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left(\mu^{-1} b_h^2 - h_h^1\right)\right\|^2 \right)$$

subject to $de_h^1 = -\gamma\left(b_h^2 - \bar{b}_h^2\right)$ and $dh_h^1 = j_h^2$

Conforming

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} j^2 - e^1\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left(\mu^{-1} b^2 - h^1\right)\right\|^2 \right)$$

subject to $\delta e^1 = -\gamma\left(b^2 - \bar{b}^2\right)$ and $\delta h^1 = j^2$

Direct

Advantages $\Longrightarrow$

- Does not require a primal-dual grid complex
- Explicit discretization of **material laws** is avoided
- Construction of a **discrete $*$ operation** not required

So, where are the least-squares?
(An answer to the $64K Question)

We start from the (fully) mimetic discrete 4-field principle

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} j_h^2 - e_h^1\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left(\mu^{-1} b_h^2 - h_h^1\right)\right\|^2 \right) \quad \text{subject to} \quad de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \text{and} \quad dh_h^1 = j_h^2$$

But, instead of using Lagrange multipliers we note that constraints can be satisfied **exactly**.

$\Rightarrow$ we can **eliminate** the variables in the **ranges** of the differential operators:

$$de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \Rightarrow \quad b_h^2 = \overline{b}_h^2 - \gamma^{-1} de_h^1 \quad \Rightarrow \quad \mu^{-1} b_h^2 - h_h^1 = \mu^{-1}\overline{b}_h^2 - (\mu\gamma)^{-1} de_h^1 - h_h^1$$

$$dh_h^1 = j_h^2 \quad \Rightarrow \quad j_h^2 = dh_h^1 \quad \Rightarrow \quad \sigma^{-1} j_h^2 - e_h^1 = \sigma^{-1} dh_h^1 - e_h^1$$

The **constrained** 4 field principle reduces to the **unconstrained** (least-squares) problem

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} dh_h^1 - e_h^1\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left((\mu\gamma)^{-1} de_h^1 + h_h^1 - \mu^{-1}\overline{b}_h^2\right)\right\|^2 \right)$$

$\Rightarrow$ **a Mimetic LSP is equivalent to a fully compatible discretization of the 4-field principle**

Where are the mixed methods?

A **fully mimetic** discretization of the semidiscrete 4-field principle

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(*_{\sigma^{-1}} j - e\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left(*_{\mu^{-1}} b - h\right)\right\|^2 \right) \quad \text{subject to} \quad de = -\gamma\left(b - \bar{b}\right) \quad \text{and} \quad dh = j$$

uses mimetic approximations for both the primal and the dual variables:

$$\Lambda^1(d,K) \times \Lambda^2(d,K) \Leftarrow \left(e_h^1, b_h^2\right) \quad \Longleftarrow \quad (e,b);(h,j) \quad \Longrightarrow \quad \left(h_h^1, j_h^2\right) \Rightarrow \Lambda^1(d,K) \times \Lambda^2(d,K)$$

and **reduces to a mimetic least-squares**. However, we can apply mimetic discretization to just one of the two pairs of variables, either the primal or the dual:

$$\Lambda^1(d,K) \times \Lambda^2(d,K) \Leftarrow \left(e_h^1, b_h^2\right) \quad \Longleftarrow \quad (e,b) \quad \Longleftarrow \quad \left(e_h^2, b_h^1\right) \Rightarrow \Lambda^2(d,K) \times \Lambda^1(d,K)$$

$$\Lambda^2(d,K) \times \Lambda^1(d,K) \Leftarrow \left(h_h^2, j_h^1\right) \quad \Longrightarrow \quad (h,j) \quad \Longrightarrow \quad \left(h_h^1, j_h^2\right) \Rightarrow \Lambda^1(d,K) \times \Lambda^2(d,K)$$

A **primal mimetic** method

A **dual mimetic** method

The primal mimetic method

We start from the primal mimetic discrete 4-field principle

$$\min \frac{1}{2}\left(\left\|\sqrt{\sigma}\left(\sigma^{-1} j_h^1 - e_h^1\right)\right\|^2 + \left\|\sqrt{\mu\gamma}\left(\mu^{-1} b_h^2 - h_h^2\right)\right\|^2\right) \text{ subject to } de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \text{ and } d^* h_h^2 = j_h^1$$

Clearly, the minimum is achieved when $j_h^1 = \sigma e_h^1$ and $h_h^2 = \mu^{-1} b_h^2$. Instead of **eliminating the constraints** now we **eliminate the functional** and obtain the discrete system

$$de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \text{and} \quad d^* \mu^{-1} b_h^2 = \sigma e_h^1$$

Using that $\left(d^* h_h^2, \hat{e}_h^1\right) = \left(h_h^2, d\hat{e}_h^1\right) + \left\langle h_t, \hat{e}_h^1 \right\rangle_{\Gamma_2} \ \forall\, \hat{e}_h^1 \in \Lambda^1(d,K)$ gives the **mixed problem**

$$de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \text{and} \quad \left(\mu^{-1} b_h^2, d\hat{e}_h^1\right) + \left\langle h_t, \hat{e}_h^1 \right\rangle_{\Gamma_1} = \left(\sigma e_h^1, \hat{e}_h^1\right) \quad \forall\, \hat{e}_h^1 \in \Lambda^1(d,K)$$

The **range** variable can be **eliminated** to obtain a **Rayleigh-Ritz** type equation

$$\gamma\left(\sigma e_h^1, \hat{e}_h^1\right) + \left(\mu^{-1} de_h^1, d\hat{e}_h^1\right) = \gamma\left\langle h_t, \hat{e}_h^1 \right\rangle_{\Gamma_1} + \gamma\left(\mu^{-1} \overline{b}_h^2, d\hat{e}_h^1\right) \qquad \forall\, \hat{e}_h^1 \in \Lambda^1(d,K)$$

It is a fully discrete version of the equivalent, **second order** eddy current equation

$$\sigma\dot{\mathbf{E}} + \nabla \times \mu^{-1} \nabla \times \mathbf{E} = 0$$

The three methods: summary

Fully mimetic

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} j_h^2 - e_h^1\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left(\mu^{-1} b_h^2 - h_h^1\right)\right\|^2 \right) \quad \text{subject to} \quad de_h^1 = -\gamma\left(b_h^2 - \bar{b}_h^2\right) \quad \text{and} \quad dh_h^1 = j_h^2$$

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} dh_h^1 - e_h^1\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left((\mu\gamma)^{-1} de_h^1 + h_h^1 - \mu^{-1}\bar{b}_h^2\right)\right\|^2 \right)$$

Primal mimetic

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} j_h^1 - e_h^1\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left(\mu^{-1} b_h^2 - h_h^2\right)\right\|^2 \right) \quad \text{subject to} \quad de_h^1 = -\gamma\left(b_h^2 - \bar{b}_h^2\right) \quad \text{and} \quad d^* h_h^2 = j_h^1$$

$$de_h^1 = -\gamma\left(b_h^2 - \bar{b}_h^2\right) \quad \text{and} \quad \left(\mu^{-1} b_h^2, d\hat{e}_h^1\right) + \left\langle h_t, \hat{e}_h^1 \right\rangle_{\Gamma_1} = \left(\sigma e_h^1, \hat{e}_h^1\right) \quad \forall\, \hat{e}_h^1 \in \Lambda^1(d,K)$$

$$\gamma\left(\sigma e_h^1, \hat{e}_h^1\right) + \left(\mu^{-1} de_h^1, d\hat{e}_h^1\right) = \gamma\left\langle h_t, \hat{e}_h^1 \right\rangle_{\Gamma_1} + \gamma\left(\mu^{-1}\bar{b}_h^2, d\hat{e}_h^1\right) \quad \forall\, \hat{e}_h^1 \in \Lambda^1(d,K) \qquad \sigma\dot{\mathbf{E}} + \nabla \times \mu^{-1} \nabla \times \mathbf{E} = 0$$

Dual mimetic

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} j_h^2 - e_h^2\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left(\mu^{-1} b_h^1 - h_h^1\right)\right\|^2 \right) \quad \text{subject to} \quad d^* e_h^2 = -\gamma\left(b_h^1 - \bar{b}_h^1\right) \quad \text{and} \quad dh_h^1 = j_h^2$$

$$dh_h^1 = j_h^2 \quad \text{and} \quad \left(\sigma^{-1} j_h^2, d\hat{h}_h^1\right) + \left\langle e_t, \hat{h}_h^1 \right\rangle_{\Gamma_2} = -\gamma\left(\mu h_h^1 + \bar{b}_h^1, \hat{h}_h^1\right) \quad \forall\, \hat{h}_h^1 \in \Lambda^1(d,K)$$

$$\gamma\left(\mu h_h^1, \hat{h}_h^1\right) + \left(\sigma^{-1} dh_h^1, d\hat{h}_h^1\right) = -\left\langle e_t, \hat{h}_h^1 \right\rangle_{\Gamma_2} + \gamma\left(\bar{b}_h^1, \hat{h}_h^1\right) \quad \forall\, \hat{h}_h^1 \in \Lambda^1(d,K) \qquad \mu\dot{\mathbf{H}} + \nabla \times \sigma^{-1} \nabla \times \mathbf{H} = 0$$

Mimetic LS = Primal + Dual Mimetic

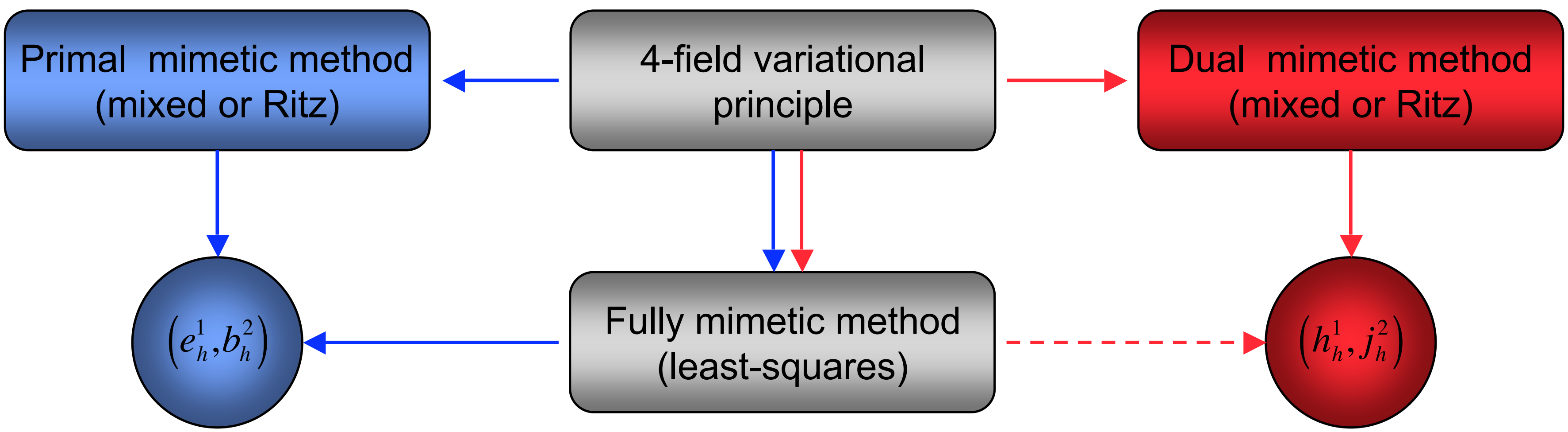
Theorem Let $\left(e_h^1, b_h^2\right), \left(h_h^1, j_h^2\right)$ be the mimetic **least-squares** solution. Then

$\left(e_h^1, b_h^2\right)$ is the solution of the **primal** mimetic method

If $\boldsymbol{b(x,0)=0}$, or we solve in frequency domain, we also have that

$\left(h_h^1, j_h^2\right)$ is the solution of the **dual** mimetic method

This means, mimetic LS is equivalent to simultaneous solution of the primal and dual methods



Proof

The first order necessary condition for the least-squares principle is

$$\left(\sigma^{-1/2}dh_h^1-\sigma^{1/2}e_h^1,\sigma^{-1/2}d\hat{h}_h^1-\sigma^{1/2}\hat{e}_h^1\right)+\left((\mu\gamma)^{-1/2}de_h^1+(\mu\gamma)^{1/2}h_h^1,(\mu\gamma)^{-1/2}d\hat{e}_h^1+(\mu\gamma)^{1/2}\hat{h}_h^1\right)$$

$$=\left(\mu^{-1}(\mu\gamma)^{1/2}\overline{b}_h^2,(\mu\gamma)^{-1/2}d\hat{e}_h^1+(\mu\gamma)^{1/2}\hat{h}_h^1\right)$$

Expand each term

$$\left(\sigma^{-1/2}dh_h^1-\sigma^{1/2}e_h^1,\sigma^{-1/2}d\hat{h}_h^1-\sigma^{1/2}\hat{e}_h^1\right)=\left(\sigma e_h^1,\hat{e}_h^1\right)+\left(\sigma^{-1}dh_h^1,d\hat{h}_h^1\right)-\left(dh_h^1,\hat{e}_h^1\right)-\left(e_h^1,d\hat{h}_h^1\right)$$

$$\left((\mu\gamma)^{-1/2}de_h^1+(\mu\gamma)^{1/2}h_h^1,(\mu\gamma)^{-1/2}d\hat{e}_h^1+(\mu\gamma)^{1/2}\hat{h}_h^1\right)=\gamma\left(\mu h_h^1,\hat{h}_h^1\right)+\gamma^{-1}\left(\mu^{-1}de_h^1,d\hat{e}_h^1\right)+\left(de_h^1,\hat{h}_h^1\right)+\left(h_h^1,d\hat{e}_h^1\right)$$

The least-squares optimality system **uncouples** into two independent equations

$$\gamma\left(\sigma e_h^1,\hat{e}_h^1\right)+\left(\mu^{-1}de_h^1,d\hat{e}_h^1\right)=\gamma\left\langle h_t,\hat{e}_h^1\right\rangle_{\Gamma_2}+\gamma\left(\mu^{-1}\overline{b}_h^2,d\hat{e}_h^1\right)\quad\forall\hat{e}_h^1\in\Lambda^1(d,K)$$

Primal mimetic

$$\gamma\left(\mu h_h^1,\hat{h}_h^1\right)+\left(\sigma^{-1}dh_h^1,d\hat{h}_h^1\right)=-\left\langle e_t,\hat{h}_h^1\right\rangle_{\Gamma 1}+\gamma\left(\overline{b}_h^2,\hat{h}_h^1\right)\quad\forall\hat{h}_h^1\in\Lambda^1(d,K)$$

If $b(x,0)=0$, or in frequency domain, then the 2nd LS equation is identical to

$$\gamma\left(\mu h_h^1,\hat{h}_h^1\right)+\left(\sigma^{-1}dh_h^1,d\hat{h}_h^1\right)=-\left\langle e_t,\hat{h}_h^1\right\rangle_{\Gamma 1}+\gamma\left(\overline{b}_h^1,\hat{h}_h^1\right)\quad\forall\hat{h}_h^1\in\Lambda^1(d,K)$$

Dual mimetic

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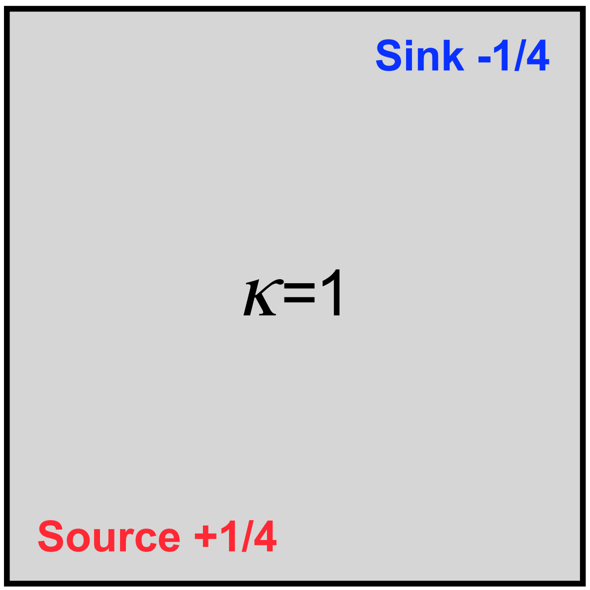
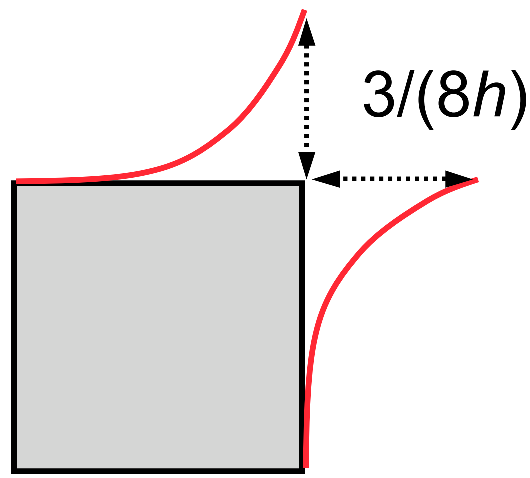
Part II

(the fun part)

Diffusion:The 5 Spot Problem

From: *T. Hughes, A. Masud and J. Wan, A stabilized mixed DG method for Darcy flow*

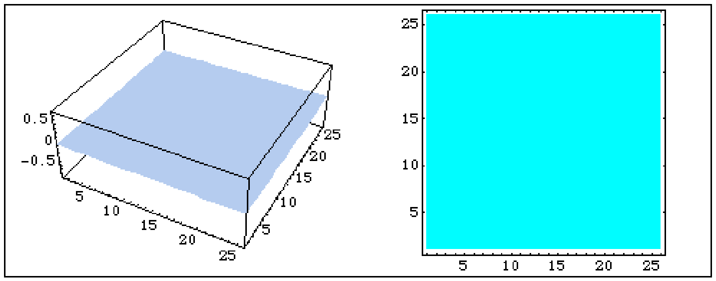




- Problem is driven by a Neumann boundary condition (normal flux)
- Source/Sink is approximated by an equivalent distribution of the normal flux
- Solved as a time-dependent problem (heat equation) using Implicit Euler
- Grid has 625 uniform quad elements

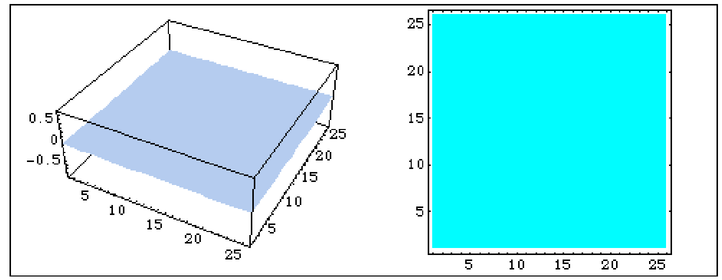
No Source Term

25x25 Uniform Elements



Mimetic LS Pressure dt=0.01, nt=100

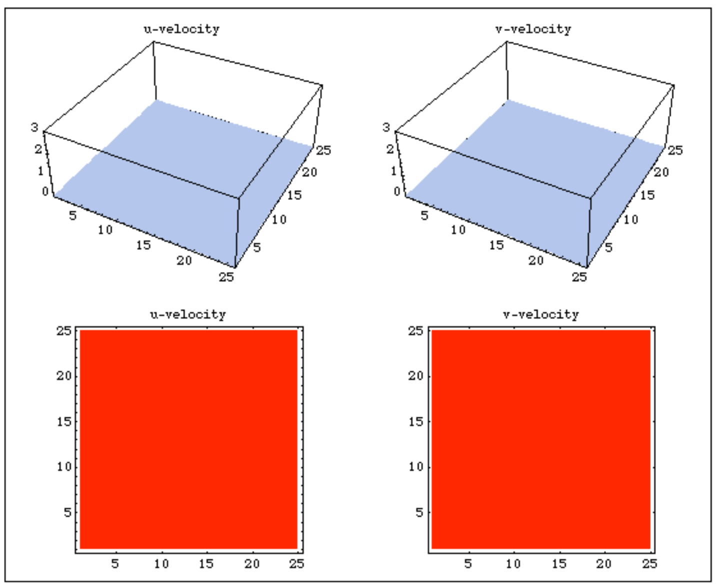
25x25 Uniform Elements



Q1-Q1 LS Pressure dt=0.01, nt=100

mimetic

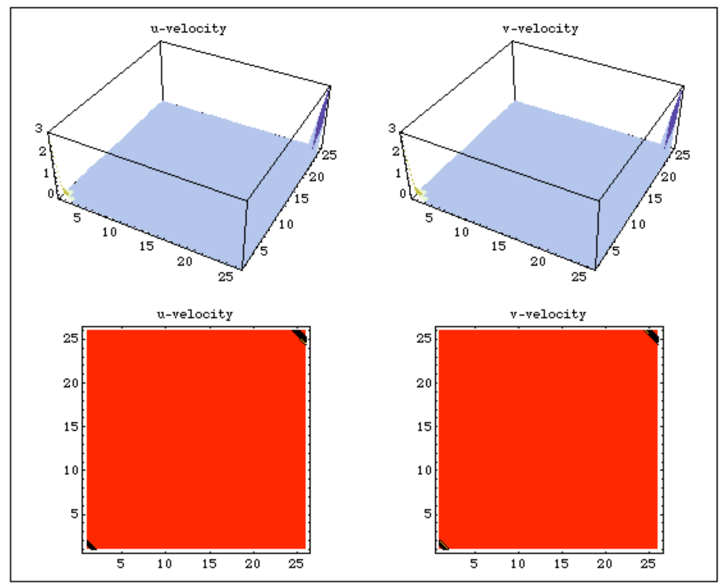
25x25 Uniform Elements



Mimetic LS Velocity. dt=0.01, nt=100

nodal

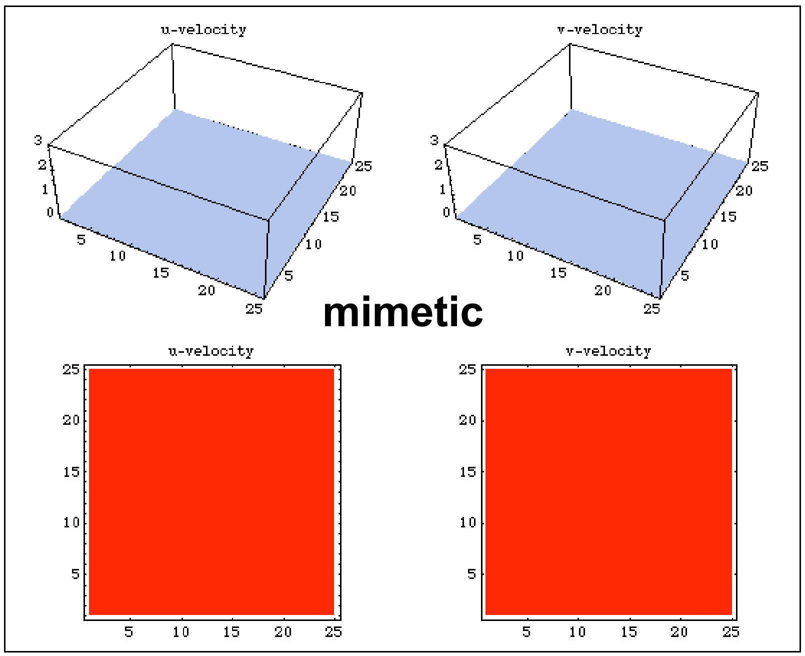
25x25 Uniform Elements



Q1-Q1 LS Velocity dt=0.01, nt=100

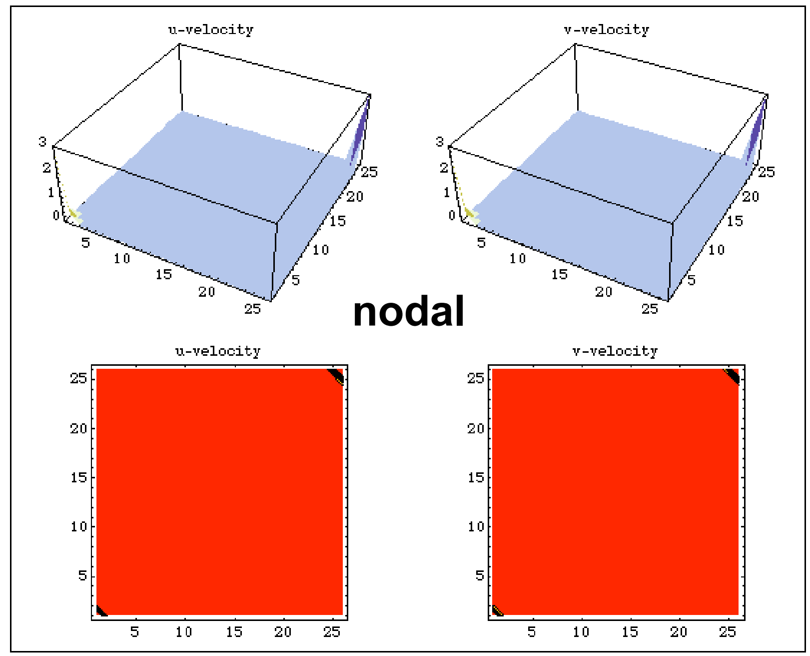
Oscillatory Source

25x25 Uniform Elements



Mimetic LS Velocity. dt=0.01, nt=100

25x25 Uniform Elements



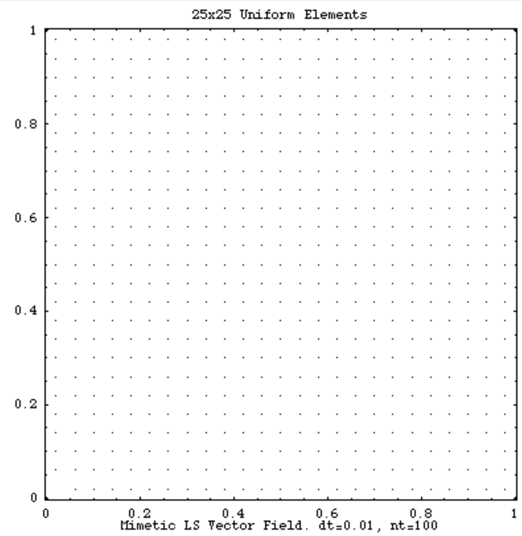
Q1-Q1 LS Velocity dt=0.01, nt=100

$$f = n\cos(\pi(n-1)x)\cos(\pi(n-1)y) \approx \frac{1}{\pi}\sqrt{\frac{\lambda}{2}}\,\varphi_{n,n}; \quad n = 25$$

$$\text{added perturbation} \approx \frac{1}{2\pi^2 n}\left|\varphi_{n,n}\right| \leq 0.002$$

Vector Field Comparison

Mimetic LS



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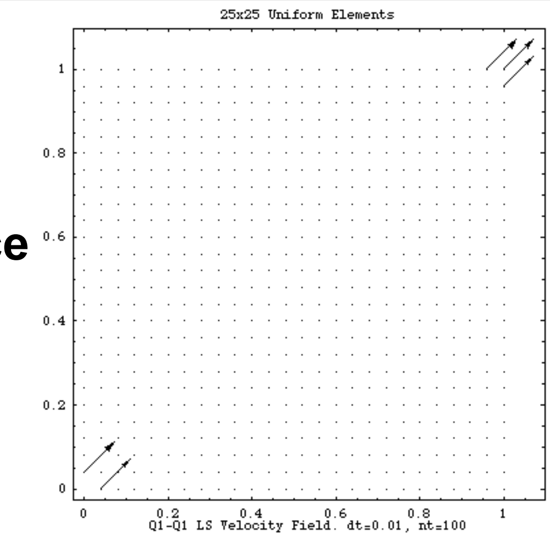
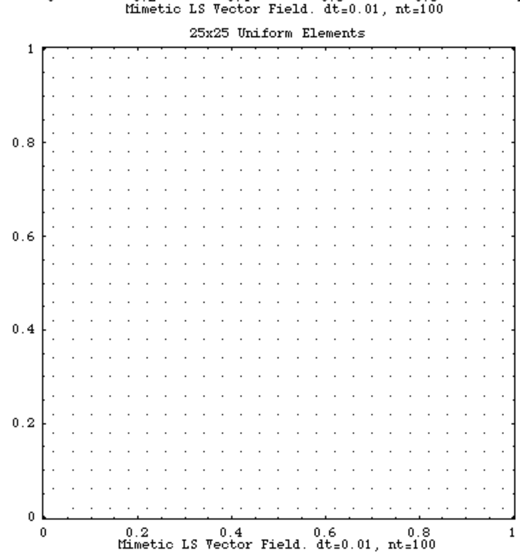


Nodal LS



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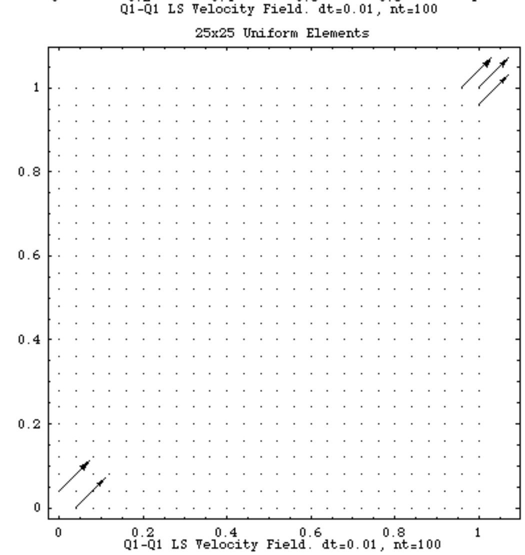
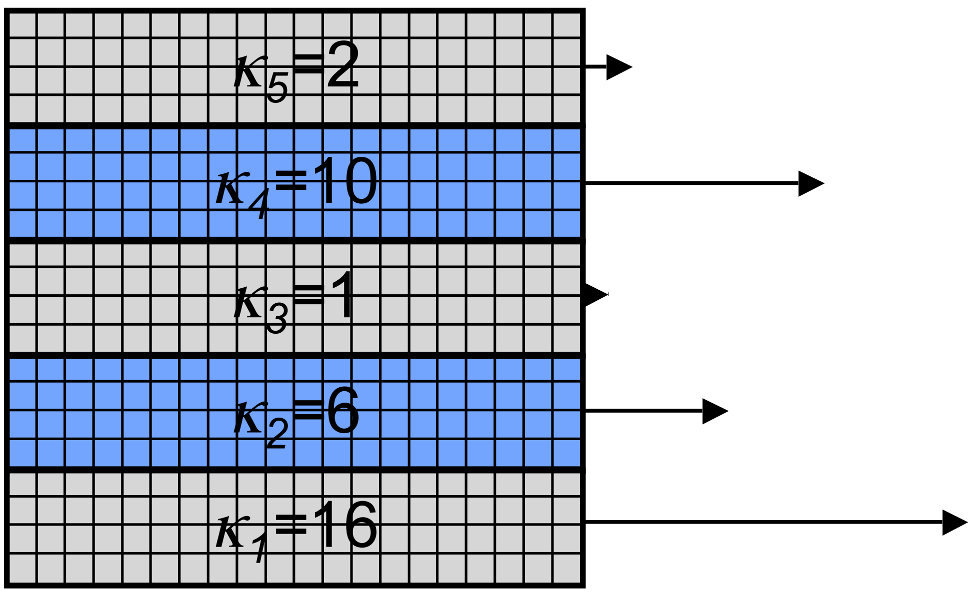


Diffusion: The 5 Strip Problem

From: *T. Hughes, A. Masud and J. Wan, A stabilized mixed DG method for Darcy flow*



Exact solution

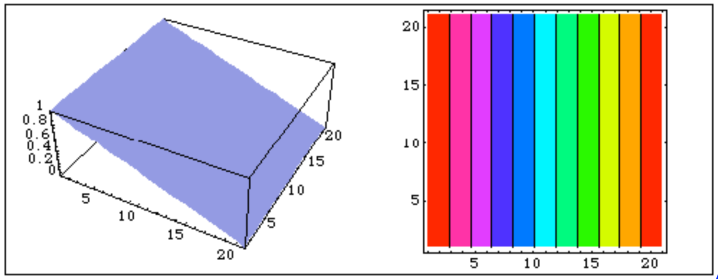
$$\phi = 1 - x;$$

$$\mathbf{u} = \begin{pmatrix} \kappa_i \\ 0 \end{pmatrix} \text{ in strip } i$$

- Problem is driven by Neumann boundary condition (normal flux)
- Solved as a time-dependent problem (heat equation) using Implicit Euler
- Grid has 400 uniform elements aligned with the interfaces between the strips

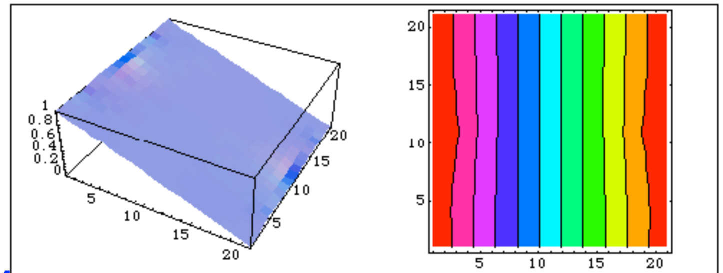
Mimetic vs. Nodal Least Squares

5 Strip Problem



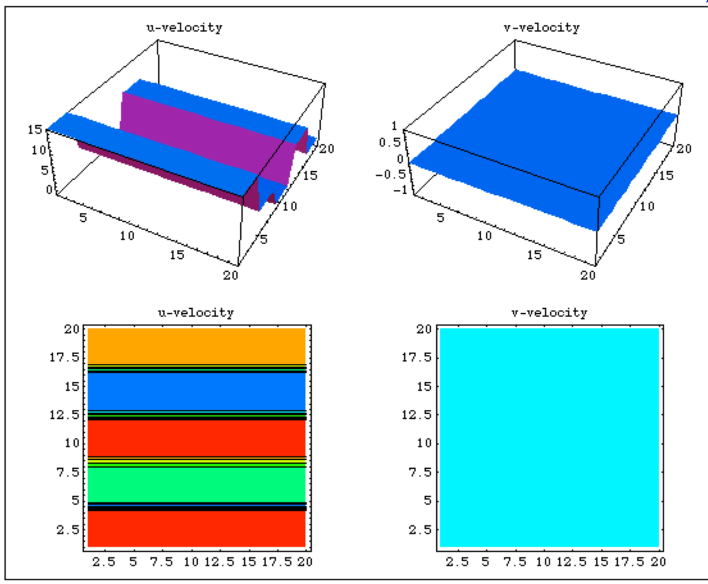
Mimetic LS Pressure

5 Strip Problem

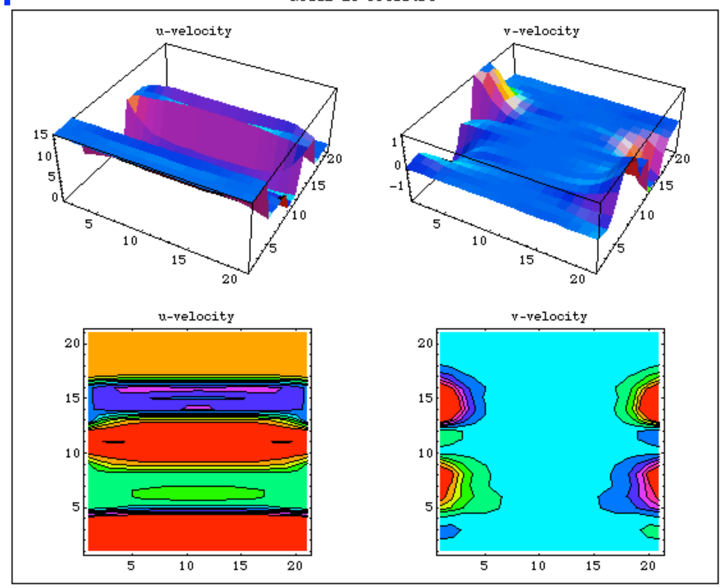


Nodal LS Pressure

Δt=0.01



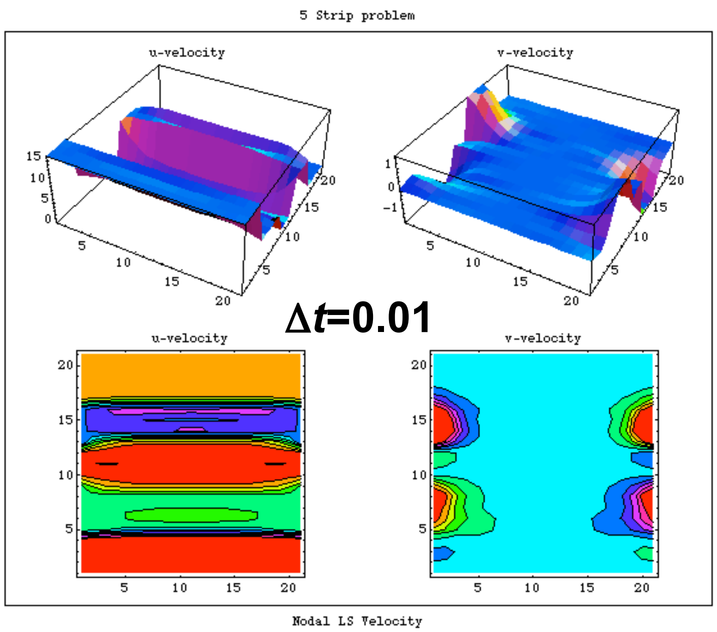
Mimetic LS Velocity

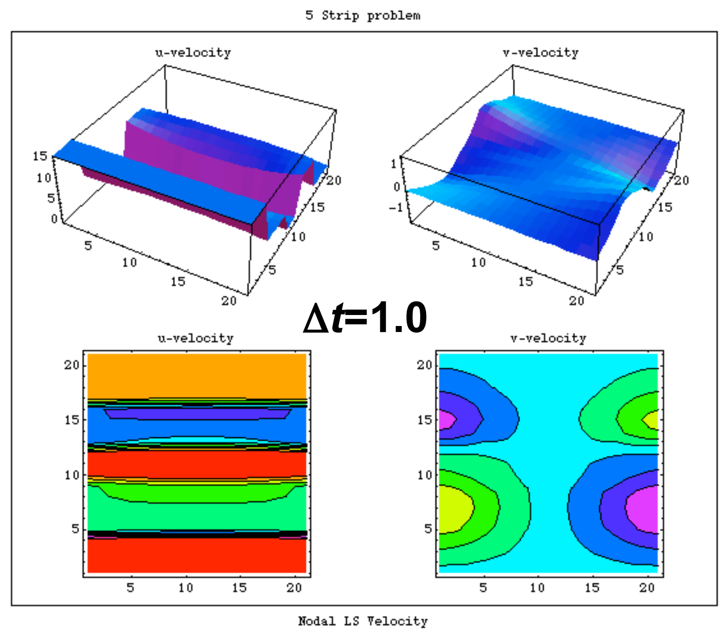


Nodal LS Velocity

	L2 Flux	H(div) Flux	L2 Scalar	H1 Scalar
Mimetic LS	0.1670E-08	0.9839E-13	0.4553E-11	0.3041E-13
Nodal LS	0.1759E+01	0.7470E+02	0.8926E-02	0.1425E+00

Nodal LS at different time steps



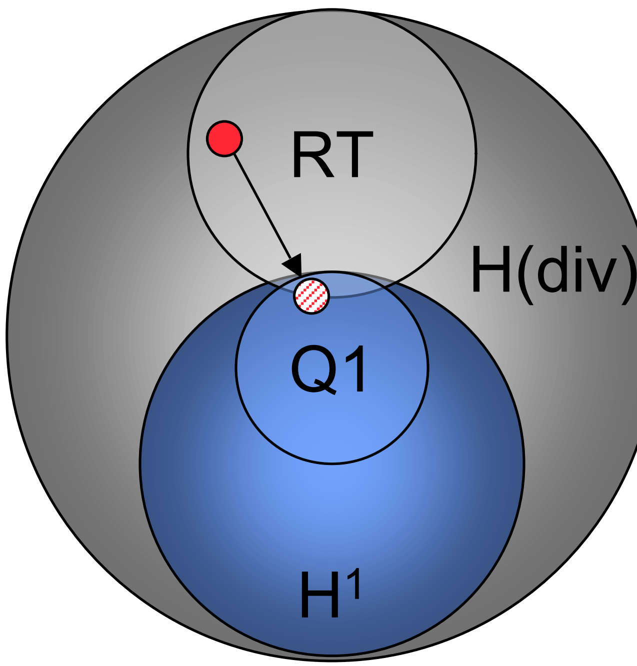


Nodal LS Solution worsens when **Δt is reduced**

	L2 Flux	H(div) Flux	L2 Scalar	H1 Scalar
Δt=1.0	0.1925E+01	0.7206E+02	0.8892E-02	0.1423E+00
Δt=0.01	0.1759E+01	0.7470E+02	0.8926E-02	0.1425E+00

Why Nodal LS fails?

Solution of the 5 strip problem belongs to the discrete space: **recovered by the mimetic LS**



Least-Squares solution is a projection onto the discrete space
$\Downarrow$
gives the best possible approximation out of that space with respect to the energy norm

Nodal Least-Squares: gives the **best energy norm** approximation of that solution out of Q1

Conclusions (I)

Mimetic Least-Squares (MLS) for 2nd order PDEs result from **weakly enforced** material laws and provide **realization** of a discrete Hodge * operator

MLS offer important advantages:

- ✓ **discrete spaces** not subject to a **joint inf-sup:** can be selected **independently**!
- ✓ **MLS** inherit the **best** computational properties of primal and dual mimetic:

  Primal → Optimal accuracy in the **primal** variable

  Dual → Optimal accuracy in the **dual** variable

- ✓ **MLS** are **locally conservative**
- ✓ **MLS** lead to **symmetric and positive definite** algebraic systems

Mimetic least-squares are an attractive alternative to mixed and finite volume schemes

Conclusions (II)

There's no free lunch: least-squares are not **immune** to compatibility:

- ✓ LS **allow** to circumvent **compatibility between the spaces**
- ☠ LS **do not allow** to circumvent **compatibility of spaces**

The latter is governed by **PDE structure** and must be respected!

References

1. P. Bochev and M. Hyman, *Principles of mimetic discretiations*, **Proc. IMA Workshop on Compatible discretizations**, Springer Verlag, To appear 2006.
2. P. Bochev, *A discourse on variational and geometric aspects of stability of discretizations*. In: 33rd **Computational Fluid Dynamics Lecture Series**, VKI LS 2003, Von Karman Institute for Fluid Dynamics
3. P. Bochev and M. Gunzburger, *Locally conservative least-squares methods for the Darcy flow*, **CMAME**, submitted
4. P. Bochev and P. Gunzburger, *Compatible discretizations of second order elliptic problems*, **Notices of the Steklov Institute**, St. Petersburgh branch, 2005
5. P. Bochev and M. Gunzburger, On least-squares finite element methods for the Poisson equation and their connection to the Dirichlet and Kelvin principles. **SIAM J. Num. Anal.**, Vol. 43/1, pp. 340-362, 2005

Related work

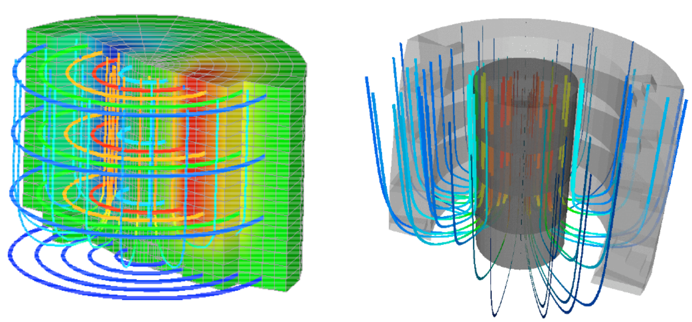
1. I. Perugia, A field-based mixed formulation for the 2D magnetostatics problem, **SINUM** 34, 1997
2. F. Brezzi, et al, A novel field-based mixed formulation of magnetostatics **IEEE MAG**-32, 1996
3. A. Bossavit, A rationale for edge elements in 3D fields computations, **IEEE MAG**-24, 1988

Magnetic Diffusion: Z-Pinch Model

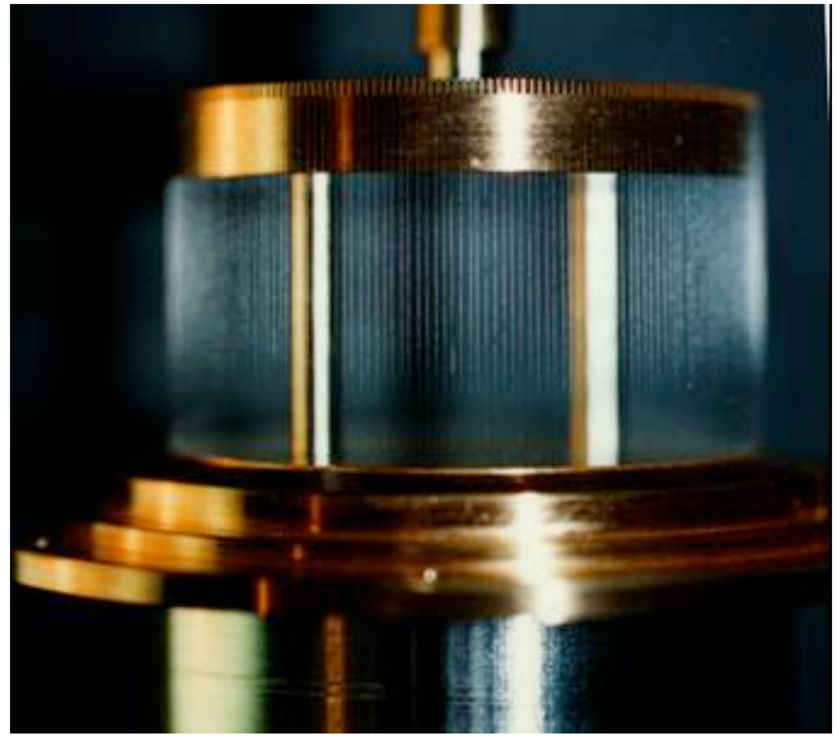
Scales:

PULSE DURATION	**10^{-9} sec**
TIME SCALE	**10^{-3} sec**
CURRENT POWER	**20×10^{6} A**
X-RAY POWER	**10^{12} W**
X-RAY ENERGY	**1.9×10^{6} J**

C. Garasi, A. Robinson

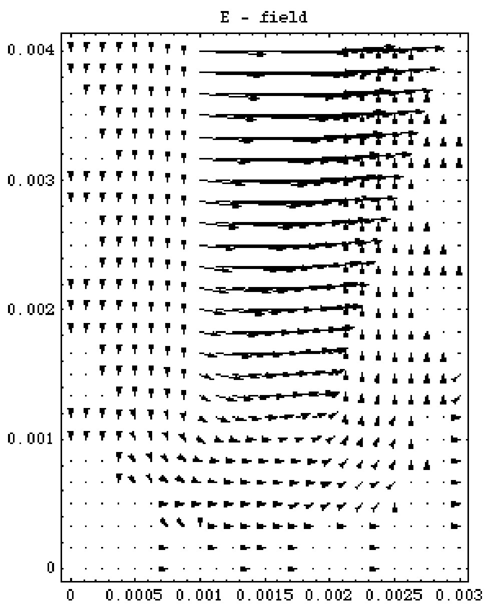


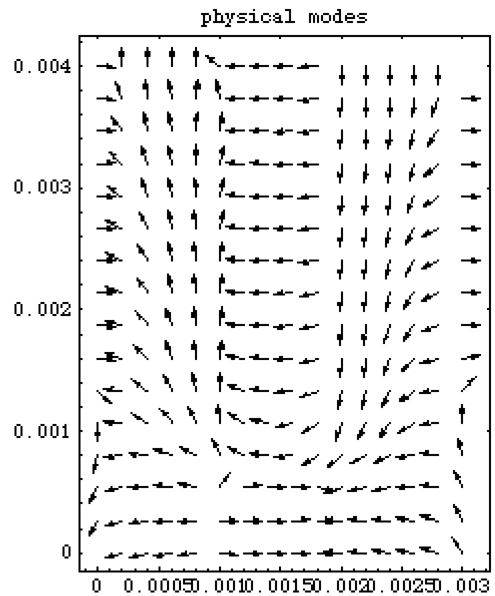
MHD MODEL
=
Hydrodynamics + Magnetic Diffusion

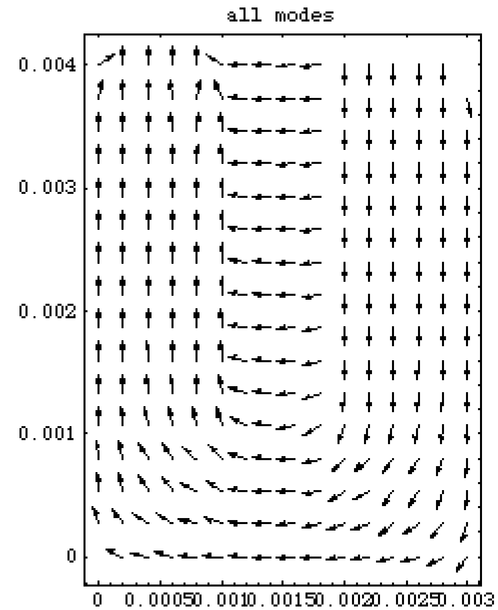


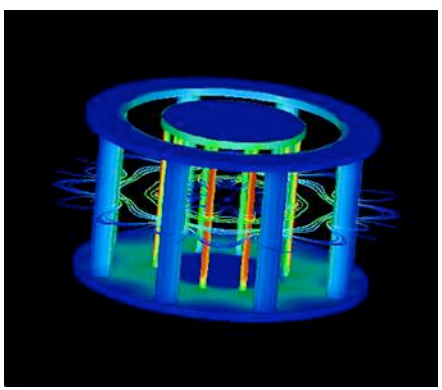
Z-machine: Electric currents are used to produce an ionized gas by vaporizing a spool-of-thread sized array of 100-400 wires of diameter ≈ 10μm

Mimetic LS vs. Nodal LS: E-field







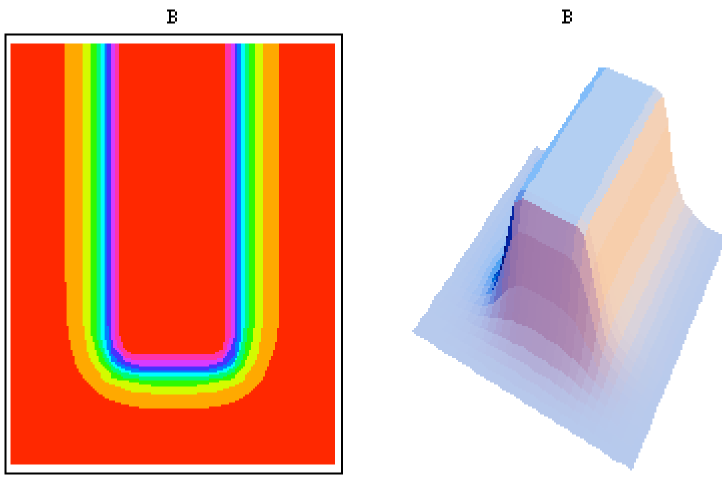


$$\nabla \times \frac{1}{\mu}\mathbf{B} = \sigma\mathbf{E} \quad \text{Ampere}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday}$$

Gap modeled as a heterogeneous conductor

Mimetic LS vs. Nodal LS: B-field



Mimetic LS

Ker(curl)={grad p}

Nodal LS

Ker(curl)={0}